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ENTANGLEMENT IN THE DICKE MODEL

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RESUMEN

Estudiamos el enredamiento cuántico en un sistema formado por N qubits en interacción con un modo electromagnético (modelo de Dicke). Presentamos resultados para el enredamiento entre dos qubits, enredamiento entre un qubit y el resto del sistema, y el enredamiento entre los qubits y el modo electromagnético. Sólo el caso de temperatura nula es analizado.

ABSTRACT

We studied the entanglement in a system of N qubits interacting with an electromagnetic mode (Dicke model). We report results about the entanglement between two qubits, the entanglemet between a qubit and the rest of the system and the entanglement between the qubits and the electromagnetic mode. The zero temperature case is considered.

INTRODUCTION

The entanglement in thermodynamic systems has been studied in spin models [1, 2, 3]. In this work, we consider a more general situation where qubits (spins) interact with a quantized radiation field, the so-called Dicke model (DM). This model can describe atoms, quantum dots, molecules or any group of identical two-level systems within a cavity.

Since the entanglement is a quantity associated with the quantum state of a system, it gives us important information over the transformations in the system's state across a phase transition.

The entanglement's behavior as reported in [1, 2, 3] has shown many surprising features. Our case it is not an exception. Nevertheless, our results present big differences as compared with those of spin systems interacting with a magnetic field.

At the present moment, there are many open questions about the entanglement in the DM. At a technical level, there are not explicit expressions to calculate the entanglement between any partitions of the system. From a physical point of view, entanglement and energy show similar features. The affirmation "The similarities between entanglement and energy appear to be more than just superficial" [1] seems to confirm itself in the present work.

DICKE MODEL OF SUPERRADIANCE

We consider a system of N identical (but distinguishible) two-level systems (qubits) interacting with a quantized radiation field in a cavity. The Hamiltonian of the model is:

$$H = aa^\dagger + \sum_{j=1}^{N} (\frac{1}{2}\epsilon\sigma_j^z + \frac{\lambda}{2\sqrt{N}}(a\sigma_j^+ + a^\dagger\sigma_j^- + a\sigma_j^- + a^\dagger\sigma_j^+)) \tag{1}$$

where, $a^\dagger$ and a are creation and annihilation operators for the cavity mode, ϵ is the ratio between the two levels energy splitting and the photon energy. In this work $\epsilon = 1$ (resonance condition), λ measures the interaction between the qubits and the field, and $\sigma_j^{x,y,z}$ are the Pauli spin matrices corresponding to the jth qubit. In addition, we have also used the ladder operators: $\sigma_j^+ = \sigma_j^x + i\sigma_j^y$ and $\sigma_j^- = \sigma_j^x - i\sigma_j^y$.

In (1) the rotating wave approximation (RWA) is obtained when the term $a\sigma_j^- + a^\dagger\sigma_j^+$ is removed from the Hamiltonian.

The Hamiltonian (1) conmutes with the parity operator Θ:

$$\Theta = e^{i\pi\hat{N}}, \qquad \hat{N} = a^\dagger a + \frac{1}{2}\sum_{j=1}^{N}\sigma^z + \frac{N}{2} \tag{2}$$

where $\hat{N}$ is the "excitation number". Θ has two eigenvalues, ± 1, depending whether the eigenvalue of $\hat{N}$ is even or odd. In RWA $\hat{N}$ is exactly conserved.

In the thermodynamic limit ($N \to \infty$) this model displays a phase transition, both at nonzero temperature and at zero temperature. The latter case is known as a *quantum phase transition* (QPT).

In the QPT the symmetry associated with the parity operator Θ is broken at $\lambda_c = 0.5$: for λ smaller than λ_c the ground-state is nondegenerate and therefore is an eigenstate of Θ. For λ larger than λ_c the ground-state is twofold degenerate and each eigenstate may not be an eigenstate of Θ. The change in the ground-state's structure can be easily detected through the mean photon number. In the more symmetric phase the mean photon number vanishes but in the less symmetric phase (the superradiant phase) it becomes different from zero, see Fig. 1.

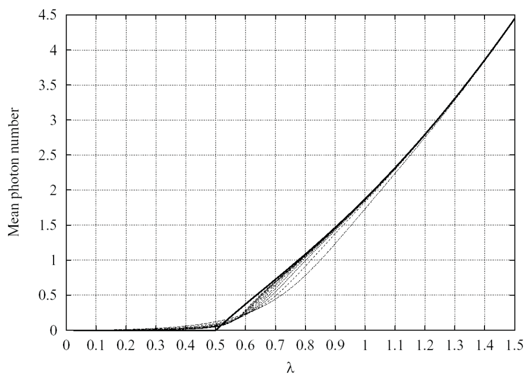


Figure 1: The mean photon number $\frac{2\langle a^\dagger a\rangle}{N}$ for $N = 2, 3, ..., 12$. The solid line corresponds to the thermodynamic limit

ENTANGLEMENT IN THE QPT

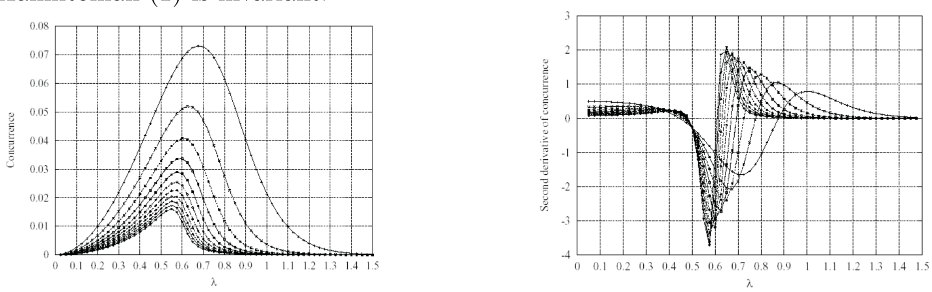
We start by considering the entanglement between two qubits in the system. In this case the amount of entanglement can be explicitly calculated using the *concurrence* C introduced by Wootters [4]. The concurrence vanishes for an unentangled state whereas $C = 1$ for a maximally entangled one.

We also used the von Neumman entropy S to calculate the entanglement between a qubit and the rest of the system as well as for the entanglement between the qubits subsystem and the electromagnetic mode. For a bipartite quantum system AB in a pure state, S is defined as [5]:

$$S = -tr(\rho_A log \rho_A) = -tr(\rho_B log \rho_B) \tag{3}$$

where ρ_A is the reduced density matrix correspondig to subsystem A and ρ_B has a similar meaning. Like C, $S = 1$ for a maximally entangled state and it vanishes for an unentangled state.

In the thermodynamic limit the entanglement between two qubits vanishes because the total entanglement is uniformly distributed between all qubits. Therefore we studied the concurrence of two qubits for finite N. In Fig. 2 it is shown the behavior of the concurrence for different N. For $\lambda = 0$ the concurrence vanishes for any N since the state corresponds to no photons in the cavity and every qubit in the ground-state, a situation which clearly corresponds to a separable state. For each N the concurrence reaches a maximum value near to λ_c. This maximun value of C goes down as the number of qubits increases. It is also evident the developing of an asymmetry in C as a function of λ for large N. By comparing Fig. 1 with 2 we find that in the region where some dashed curves in Fig. 1 smoothly agree with the solid curve, the corresponding concurrence vanishes in Fig. 2. In these regions, the behavior of the system in the thermodynamic limit is well reproduced by only a few qubits interacting with the electromagnetic mode. In Fig. 2 we also can see the behavior of the second derivative of the concurrence. The crossing in λ_c has also been seen in the second derivative of the ground-state energy [6] and it suggests the existence of finite-size scaling transformations under which the Hamiltonian (1) is invariant.

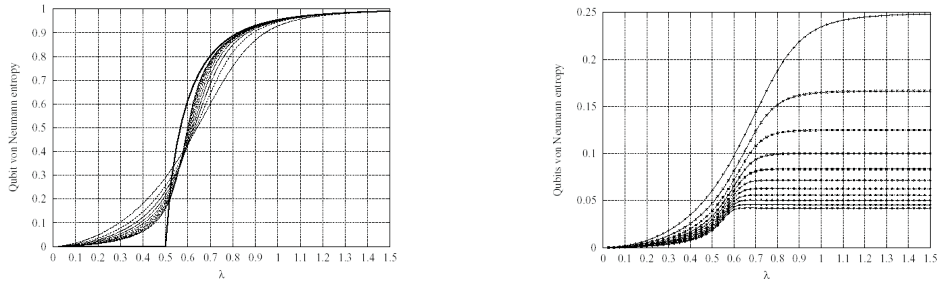


(a) The highest curve corresponds to $N = 2$ and the others correspond to $N = 3, 4, ..., 12$ in descending order.

(b) The lowest curve corresponds to $N = 2$ and the others correspond to $N = 3, 4, .., 12$ in ascending order.

Figure 2: Concurrence between two qubits and its second derivative for the Dicke model

In Fig. a the entanglement between a qubit and the rest of the system (the remainder of the qubits and the electromagnetic mode) is shown. We used the von Neumman entropy since the ground-state is nondegenerate for finite N. Again, we can see the agreement with the thermodynamic limit for only a few qubits. In the more symmetric phase the entanglement vanishes while in the superradiant phase it increases from zero to its maximum value. This entanglement measure shows a behavior similar to an order parameter. Fig. b depicts the entanglement



(a) The solid curve corresponds to the thermodynamic limit result. Dashed curves correspond to $N = 2, 3, ..12$.

(b) The highest curve corresponds to $N = 2$ and the others correspond to $N = 3, 4, ..., 12$ in descending order.

Figure 3: (a) Entanglement between a qubit and the rest of the system, (b) entanglement between the qubits and the electromagnetic mode

between the N qubits subsystem and the electromagnetic mode. In this case we don't have a theoretical result in the thermodynamic limit but we conjecture that there is no entanglement between the qubits and the electromagnetic mode in this limit because the maximum values of the curves decay as $\frac{1}{2N}$.

CONCLUSIONS

We have investigated the entanglement across the QPT in the Dicke model of superradiance. The concurrence between two qubits vanishes in the thermodynamic limit but it becomes nonzero for finite N. The second derivative of concurrence for different N shows a crossing in λ_c like the second derivative of the ground-state energy, suggesting a finite-size scaling for such a highly connected system. The entanglement between a qubit and the rest of the system behaves like an order parameter. The entanglement between the qubits and the electromagnetic mode seems to vanish in the thermodynamic limit.

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