

**THE VAN CITTERT-ZERNIKE THEOREM: A FRACTIONAL ORDER
FOURIER TRANSFORM POINT OF VIEW**

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ABSTRACT

The transfer of the coherence by a multielement optical system is studied, within the framework of the fractional Fourier transform. The method relies on the fundamental fact that the coherence is transformed by a linear filter univocally connected to the filter acting on the electric field. The results show that, the Van Cittert-Zernike theorem can be expressed in terms of the fractional order Fourier transforms of the mutual intensities over spherical surfaces M_1 and M_2 ; which allows, among others, to situate this theorem in a quite general context.

Keywords: Fractional Fourier transform, Van Cittert-Zernike theorem, Optical transformations, wave propagation.

1. INTRODUCTION

The α th order fractional Fourier transform \mathfrak{F}^α of the function $U_A(u, v)$ is defined for $0 < |\alpha| < 2$ as:

$$\mathfrak{F}^\alpha [U_A(u, v)] = \frac{\exp\left[i\left(\frac{\pi}{2} - \alpha\right)\right]}{2\pi \sin \alpha} \exp\left[\frac{-i(\xi^2 + \eta^2)}{2 \tan \alpha}\right] \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[\frac{-i(u^2 + v^2)}{2 \tan \alpha}\right] \exp\left[\frac{i(\xi u + \eta v)}{2 \sin \alpha}\right] [U_A(u, v)] du dv \tag{1}$$

Some essential properties of the fractional Fourier transform have already been presented in Refs. 1-5. Optical implementations, signal-processing applications, the transform as a tool for analyzing and describing optical systems and experimental results, may be found in Refs. 2-4. In Refs. 6-12, the statistical properties of light in terms of the fractional Fourier transform may be found. In all of the references mentioned above, the transfer of the coherence by a multielement optical system and the Van Cittert-Zernike theorem are ignored and the propagation of mutual intensity expressed in terms of the fractional Fourier transform is only assumed. The present paper extends these results to the Van Cittert-Zernike.

II. COHERENCE THEORY OF MULTIELEMENT OPTICAL SYSTEMS

The mutual coherence relative to the couple of points (\vec{S}_1, \vec{S}_2) of the surface M_1 is defined by:

$$\Gamma(\bar{S}_1, \bar{S}_2, \tau) = E[V(\bar{S}_1, t)V^*(\bar{S}_2, t + \tau)] \quad (2)$$

Where $V(\bar{S}_1, t)$ is the complex amplitude of the optical field. The intensity profile can be obtained by $I_{M_1}(\bar{S}) = \Gamma(\bar{S}, \bar{S}, 0)$. According to the theorem Bernstein, Wiener, Kinchine (BWK); the cross spectral density (mutual intensity) is defined by:

$$J(\bar{S}_1, \bar{S}_2, \nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle V(\bar{S}_1, t)V^*(\bar{S}_2, t + \tau) \rangle \exp(2i\pi\nu\tau) d\tau \quad (3)$$

The angular brackets denote time average and the asterisk denotes the complex conjugate. Under the condition of paraxial approximation, the propagation formula of the mutual intensity is given by:

$$J(\bar{\rho}_1, \bar{\rho}_2, \nu) = \frac{1}{\lambda_0^2 B^2} \exp\left[\frac{ikD}{2B}(\bar{\rho}_1^2 - \bar{\rho}_2^2)\right] \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J_0(\bar{S}_1, \bar{S}_2, \nu) \exp\left[\frac{ikA}{2B}(\bar{S}_1^2 - \bar{S}_2^2)\right] \exp\left[\frac{-ik}{B}(\bar{\rho}_1 \cdot \bar{S}_1 - \bar{\rho}_2 \cdot \bar{S}_2)\right] d^2\bar{S}_1 d^2\bar{S}_2 \quad (4)$$

Where $ABCD$ is a arbitrary lossless optical system characterized by their matrices, λ_0 is the wavelength; $\bar{S}_1(u_1, v_1)$, $\bar{S}_2(u_2, v_2)$ and $\bar{\rho}_1(\xi_1, \eta_1)$, $\bar{\rho}_2(\xi_2, \eta_2)$ are transverse coordinates on spherical surfaces M_1 and M_2 , ν is the light frequency.

III. THE VAN CITTERT-ZERNIKE THEOREM

After a little algebra; the transfer of the coherence by a multielement optical system (equation (4)) can be written in a considerably simpler manner in terms of the fractional order Fourier transform given by:

$$J(\xi_1, \xi_2, \eta_1, \eta_2; \nu) = \frac{2\pi \sin \alpha}{\lambda_0^2 B^2} \exp\left[\frac{i\pi[(\xi_1^2 - \xi_2^2) + (\eta_1^2 - \eta_2^2)]}{R_2 \lambda_0}\right] \exp\left[-i\left[\frac{\pi}{2} - \alpha\right]\right] \mathfrak{F}^\alpha[J_0(u_1, u_2, v_1, v_2; \nu)] \quad (5)$$

Where:

$$R_2 = \frac{-AB\left(1 - \frac{B}{R_1}\right)}{AD\left(1 - \frac{B}{R_1}\right) - \cos^2 \alpha} \quad (6)$$

A fractional Fourier transform relation \mathfrak{F}^α of order n_α between the mutual intensity $J_0(u_1, u_2, v_1, v_2; \nu)$ over spherical surface M_1 with radius R_1 and the mutual intensity $J(\xi_1, \xi_2, \eta_1, \eta_2; \nu)$ over spherical surface M_2 with radius R_2 , can be obtained with $\alpha = \frac{\pi}{2} n_\alpha$; (α real parameter). The fundamental result consists in that with all generality the mutual intensity over spherical surface M_1 and the mutual intensity over

spherical surface M_2 are fractional order Fourier transforms each one of the other one; it contains effectively as particular cases all results knew early: narrow- band radiation, which allows, among others, to situate Zernike theorem in a quite general context. The goal of this letter is show that, the Van Cittert-Zernike theorem can be expressed in terms of the fractional order Fourier transforms of the mutual intensities over spherical surfaces. If we normalize equation (5), we obtain the following expression for the (equal-time) complex degree of coherence:

$$j(\xi_1, \xi_2, \eta_1, \eta_2; \nu) = \frac{1}{[I(\bar{S}_1)]^{\frac{1}{2}} [I(\bar{S}_2)]^{\frac{1}{2}}} \frac{2\pi \sin \alpha}{\lambda_0^2 B^2} \exp \left[\frac{i\pi [(\xi_1^2 - \xi_2^2) + (\eta_1^2 - \eta_2^2)]}{R_2 \lambda_0} \right] \exp \left[-i \left[\left(\frac{\pi}{2} - \alpha \right) \right] \right] \mathfrak{F}^\alpha [J_0(u_1, u_2, v_1, v_2; \nu)] \quad (7)$$

The formula (7) is the mathematical formulation of the generalized Van Cittert-Zernike theorem. It expresses that the equal time degree of coherence at two points in the field generated by a spherical, source in terms of the fractional order Fourier transform of the mutual intensity of the spherical source and the intensity at the two field points. In incoherent light; the particular case of a narrow band radiation:

$$J_0(u_1, u_2, v_1, v_2; \nu) = J_0(\bar{S}_1, \bar{S}_2; \nu) = J_0(\bar{S}_1; \nu) \delta(\bar{S}_1 - \bar{S}_2) \quad (8)$$

The presence of the delta function on the right-hand side of equation (8) express the fact that any two elements of the source are assumed to be mutually uncorrelated:

$$J(\xi_1, \xi_2, \eta_1, \eta_2; \nu) = \frac{2\pi \sin \alpha}{\lambda_0^2 B^2} \exp \left[\frac{i\pi [(\xi_1^2 - \xi_2^2) + (\eta_1^2 - \eta_2^2)]}{R_2 \lambda_0} \right] \exp \left[-i \left[\left(\frac{\pi}{2} - \alpha \right) \right] \right] \mathfrak{F}^\alpha [J_0(\bar{S}_1; \nu) \delta(\bar{S}_1 - \bar{S}_2)] \quad (9)$$

Then:

$$J(\xi_1, \xi_2, \eta_1, \eta_2; \nu) = \frac{2\pi \sin \alpha}{\lambda_0^2 B^2} \exp \left[\frac{i\pi [(\xi_1^2 - \xi_2^2) + (\eta_1^2 - \eta_2^2)]}{R_2 \lambda_0} \right] \exp \left[-i \left[\left(\frac{\pi}{2} - \alpha \right) \right] \right] \mathfrak{F}^\alpha [J_0(\bar{S}_2; \nu)] \quad (10)$$

The (equal-time) complex degree of coherence of the field generated by our spatially incoherent spherical source is:

$$j(\xi_1, \xi_2, \eta_1, \eta_2; \nu) = \frac{1}{[I(\bar{S}_1)]^{\frac{1}{2}} [I(\bar{S}_2)]^{\frac{1}{2}}} \frac{2\pi \sin \alpha}{\lambda_0^2 B^2} \exp \left[\frac{i\pi [(\xi_1^2 - \xi_2^2) + (\eta_1^2 - \eta_2^2)]}{R_2 \lambda_0} \right] \exp \left[-i \left[\left(\frac{\pi}{2} - \alpha \right) \right] \right] \mathfrak{F}^\alpha [I_0(\bar{S}; \nu)] \quad (11)$$

The degree of coherence of the output beam can be easily calculated in terms of the fractional order Fourier transform of the intensity of the radiation of the M_1 .

In many cases of practical interest the mutual intensity $J_0(u_1, u_2, v_1, v_2; \nu)$ is over planar surface M_1 and the mutual intensity $J(\xi_1, \xi_2, \eta_1, \eta_2; \nu)$ is over planar

surface M_2 , this particular situation, can be obtained only if $R_1 \rightarrow \infty$ and the elements of the ray transfer matrix can be expressed as $A = D = \cos \alpha$.

IV. THE FAR-ZONE FORM OF THE VAN CITTERT-ZERNIKE THEOREM

In the particular situation when $\alpha = \frac{\pi}{2}$ and $R_1 = L$, then $R_2 = -L$ (free space propagation); or if treat the field points $P_j(\vec{\rho}, \nu)$ situated in the far-zone of the source:

$$J(\xi_1, \xi_2, \eta_1, \eta_2; \nu) = \frac{1}{\lambda_0^2 L^2} \mathfrak{F}[J_0(u_1, u_2, v_1, v_2; \nu)] \quad (12)$$

A Fourier transform relation \mathfrak{F} between the mutual intensity $J_0(u_1, u_2, v_1, v_2; \nu)$ over spherical surface M_1 with radius $R_1 = L$ and the mutual intensity over spherical surface M_2 with radius $R_2 = -L$, and separated by distance L can be obtained. In this particular situation, wave fields do not distinguish between exact and paraxial evolution (far-zone form of the Van Cittert-Zernike theorem).

In incoherent light; the particular case of a narrow band radiation, if treat the field points $P_j(\vec{\rho}, \nu)$ situated in the far-zone of the source, and free space propagation ($B = L$); according to equation (11):

$$j(\xi_1, \xi_2, \eta_1, \eta_2; \nu) = \frac{1}{[I(\bar{S}_1)]^{\frac{1}{2}} [I(\bar{S}_2)]^{\frac{1}{2}}} \frac{2\pi}{\lambda_0^2 L^2} \mathfrak{F}[I_0(\bar{S}; \nu)] \quad (13)$$

We concluded that the degree of coherence of the output beam can be easily calculated in terms of Fourier transform of the intensity of the radiation of the M_1 planar surface.

SUMMARY AND CONCLUSION

The mutual intensity distribution is one of the most common ways of characterizing the coherence of a wave field; we showed that the transfer of the coherence by a multielement optical system can be adequately expressed in terms of the fractional Fourier transform, this new formulation allows the extension of the Zernike theorem to cover the passage of all coherent or partially coherent propagation wave fields, and finally, we solve exactly the problem of the passage of such fields through arbitrary lossless optical system characterized by their $ABCD$ matrices.

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