Revista Colombiana de Física,Vol. 41, No.2, Abril 2009

Reduced States Statistics and Eentanglement Statistics in 2×2^N Systems

J. R. Martínez [a], K. M. Fonseca [a]

[a]Grupo de Óptica e Información Cuántica - GOIC, Universidad Nacional de Colombia, Bogotá - Colombia

Recibido 23 de Oct. 2007; Aceptado 6 de Mar. 2009; Publicado en línea 30 de Abr. 2009

Resumen

Se estudia un sistema, al cual denominamos universo, dado por un bit cu'antico y un entorno formado por N qubits. El estado del universo se supone puro y totalmente aleatorio, con distribuci'on uniforme. Bajo esas condiciones se calculan las distribuciones de probabilidad para la matriz densidad reducida, que caracteriza el estado del bit cu'antico, y de la concurrencia, que proporciona una medida del enredamiento entre el sistema y el entorno. El valor medio de la concurrencia tiende a uno, su m'aximo valor posible, a medida que el n'umero de qubits N aumenta, y su desviaci'on disminuye exponencialmente con N, produci' endose as'i el fen'omeno de concentraci'on de la medida. De igual manera, se encuentra que el estado del bit cu' antico tiende al m'aximamente enredado a medida que N crece.

***Palabras Clave*:**Enredamiento genérico, concentración de la medida.

Abstract

We examine a system, comprising a qubit and a N-qubit environment. The state of the universe is assumed to be pure and completely random (uniform distribution). Under these conditions we find the probability density to obtain a particular reduced density matrix for the system and the probability density to have a particular value of entanglement between system and environment as measured by concurrence. The average value of the concurrence is shown to approach one, the maximum possible value, at an exponential rate in the number of qubits N. The deviation from the average value, on the other hand, diminishes exponentially with N. Thus, concurrence displays the phenomenon of concentration of measure. Moreover, the distance between the typical qubit state and the maximally mixed state also decreases exponentially in the number of qubits of the environment.

***Keywords*:** Generic Entanglement, concentration of measure, qubits

1. Introducción

Entanglement, the distinctive feature of quantum mechanics according to Schrödinger and one of the most important resources for quantum information processing which measures the nonclassical correlations between two or more systems, is a quantity rather difficult to characterize, except in special cases. In the last few years the statistical approach to entanglement has become not only a viable alternative and a very active field of research but also has provided new foundations for quantum statistical mechanics[1].

In this paper we analyze the entanglement between a qubit and its N-qubit environment when the total state is a pure state $|\Psi\rangle$. It has been shown that von Neumann entropy of the qubit density operator ρ and concurrence

[2], given respectively by

$$E(\Psi) = H(\hat{\rho}) = -\mathrm{tr}(\hat{\rho}\ln\hat{\rho}), \tag{1a}$$

$$C(\Psi) = \sqrt{2\,(1-\mathrm{tr}\hat{\rho}^2)}. \tag{1b}$$

are good measures of this entanglement. In section 2 the probability density to obtain a particular reduced density matrix of the system, the probability distribution function of the concurrence and the mean entropy of the system are calculated. A statistic analysis of the entanglement is done in section 3, where generic entanglement and concentration of measure are shown to happen. Some conclusions are drawn in section 4.

2. Probability Distribution Function for the Reduced State

We consider a system composed of $N+1$ qubits and chose one of them as our system of interest. We find the probability distribution function of obtaining a particular density operator for this qubit, assuming that the state of the universe ($N+1$ qubits) is considered to be in a pure state, and that the probability density for each total pure state is constant (Haar's measure). Since any density operator ρ of a qubit can be written as $\rho = \frac{1}{2}\left(\mathbb{I} + \boldsymbol{S}\cdot\boldsymbol{\sigma}\right)$, where the components of the Bloch vector $\boldsymbol{S}$ are the expected values of the Pauli matrices, an arbitrary density operator is univocally determined by its Bloch vector. The probability density of obtaining a particular density operator is given by

$$p(\mathbf{S}) = \frac{\int_{\Psi\in\mathcal{C}_R} \delta(\langle\hat{\sigma}_x\rangle - S_x)\ \delta(\langle\hat{\sigma}_y\rangle - S_y)\ \delta(\langle\hat{\sigma}_z\rangle - S_z)\ d^2\Psi}{\int_{\Psi\in\mathcal{C}_R} d^2\Psi}, \tag{2}$$

where $\mathbf{S} = (S_x, S_y, S_z)\mathcal{C}_R$ is the subset of $\mathcal{H}_R$, the Hilbert space of the universe (of dimension $2^{N+1} = 2\times 2^N$) corresponding to normalized states and $\langle\hat{\sigma}_i\rangle$ is the mean value of the "pseudospin" operator $\hat{\sigma}_i$ of the system (any qubit of the universe). Direct calculation of the integral (2) provides

$$p(\boldsymbol{S}) = A_N\ (1-S^2)^{2^N-2}, \tag{3}$$

where $A_N = \frac{\Gamma(2^N+1/2)}{\pi^{3/2}\,\Gamma(2^N-1)}$, and $\Gamma(z)$ is the Gamma function. The expression (3) shows that if we take one qubit out of a system of $N+1$ qubits in a completely random pure state, it is described by a Bloch vector $\boldsymbol{S}$ with a probability density that i) depends only on the norm of the Bloch vector S, and ii) in the limit $N\to\infty$ is different from zero only for the maximally mixed state $S=0$ (see also Figure 1a). Since the distribution function $p(\boldsymbol{S})$ given by (3) depends only of the norm of the Bloch vector, a new probability function $\mathrm{p}(S)$, the probability density that a qubit out of an (N+1)-qubit system, prepared in a pure state, is in a state represented by a Bloch vector with norm S, can be defined as

$$\mathrm{p}(S) = 4\pi S^2 p(\boldsymbol{S}) = A_d\, S^2(1-S^2)^{d-2}, \tag{4}$$

where we have defined $d = 2^N$ and $A_d = \frac{4\,\Gamma(d+1/2)}{\sqrt{\pi}\,\Gamma(d-1)}$. The most probable state Bloch vector norm S goes from 1, in a 2-qubit system, to zero as the number of qubits grow (see Figure 1b). The probability density $\mathrm{p}(S)$, for more than 2 qubits, is zero for $S=0$, has a maximum for

$$S_M = \frac{1}{\sqrt{d-1}},$$

$$\mathrm{p}(S_M) = \frac{4\,\Gamma(d+1/2)}{\sqrt{\pi}\,\Gamma(d)}\left(1-\frac{1}{d-1}\right)^{d-2} \xrightarrow[d\gg 1]{} \frac{4}{e}\sqrt{\frac{d}{\pi}}, \tag{5}$$

and goes down to zero again at $S=1$. The distance between S_{I-} and S_{I+}, the inflection points of the probability density,

$$S_{I\pm} = \sqrt{\frac{5d-8\pm\sqrt{(d-2)(17d-26)}}{2(2d-3)(d-1)}} \xrightarrow[d\gg 1]{} \frac{1}{2\sqrt{d}}\sqrt{5\pm\sqrt{17}}, \tag{6}$$

define a characteristic width of the probability distribution. Equations (5) and (6) shows that the probability density is concentrated around a peak of height of the order of $\sqrt{d}$ and a width of the order of $1/\sqrt{d}$, roughly the distance of the peak from the origin $S=0$. A similar analysis can be made employing the mean value of S and its deviation,

$$\overline{S} = \int_0^1 S\,\mathrm{p}(S)\,dS = \frac{2\,\Gamma(d+1/2)}{\sqrt{\pi}\,\Gamma(d+1)} \xrightarrow[d\gg 1]{} \sqrt{\frac{8/\pi}{2d+1}}, \tag{7a}$$

$$\Delta S = \sqrt{\overline{S^2} - \overline{S}^2} \tag{7b}$$

$$= \sqrt{\frac{3}{2d+1} - \frac{4\,\Gamma(d+1/2)^2}{\pi\,\Gamma(d+1)^2}} \xrightarrow[d\gg 1]{} \sqrt{\frac{3-8/\pi}{2d+1}}. \tag{7c}$$

3. Entanglement Statistics

The concurrence can be calculated from definition (1b), $C(S) = \sqrt{1-S^2}$, which establishes a one-to-one relationship between the concurrence and the states

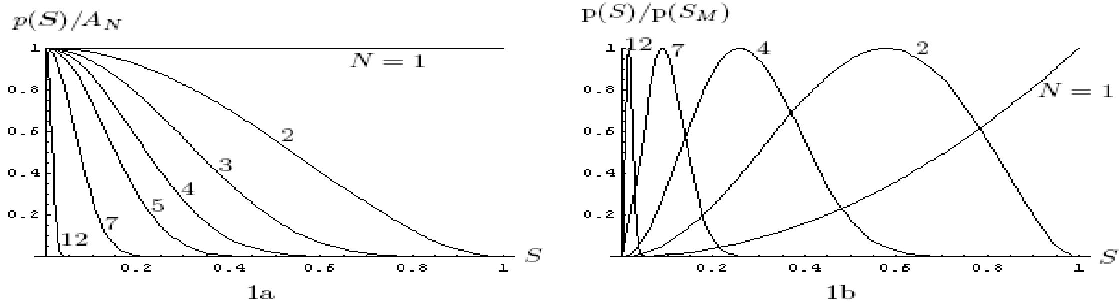

Figura 1. Probability distribution functions for a qubit in a (N+1)-qubit system in a completely random pure state. Figure 1a shows the probability density for a state with Bloch vector $\boldsymbol{S}$ and Figure 1b shows the probability density for a state with Bloch vector norm S.

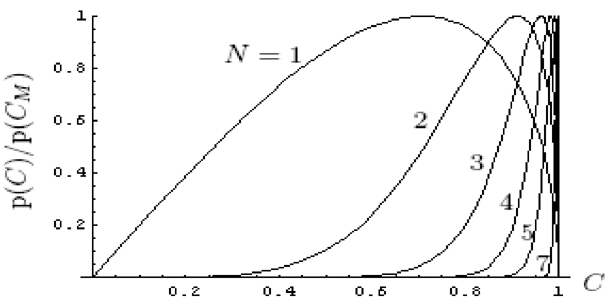

Figura 2. Probability distribution function of the qubit-environment concurrence.

with norm S in the Bloch sphere. Using the distribution function for S, given by equation (4), the distribution function for the concurrence, depicted in figure 2, is found to be

$$\mathrm{p}(C) = A_d \sqrt{1 - C^2}\, C^{2d-3}. \tag{8}$$

The most probable concurrence,

$$C_M = \sqrt{\frac{2d-3}{2d-2}} \xrightarrow[d \gg 1]{} 1 - \frac{1}{4d} \tag{9}$$

tends to 1 as $N \to \infty$. When there are many qubits in a total pure state they are most probably highly entangled, and when $N \longrightarrow \infty$ the unique possible state is the maximally entangled state, as shown in Figure 2. The same conclusions can be reach by finding the mean concurrence and its deviation

$$\overline{C} = \int_0^1 C\, \mathrm{p}(C) dC = \frac{\Gamma(d - \frac{1}{2})\Gamma(d + \frac{1}{2})}{\Gamma(d-1)\Gamma(d+1)} \xrightarrow[d \gg 1]{} 1 - \frac{3}{4d}, \tag{10a}$$

$$\Delta C = \sqrt{\frac{\Gamma(d - \frac{1}{2})^2 \Gamma(d + \frac{1}{2})^2}{\Gamma(d-1)^2 \Gamma(d+1)^2} - \frac{2d-2}{2d+1}} \xrightarrow[d \gg 1]{} \frac{\sqrt{3/8}}{d}. \tag{10b}$$

The von Neumann entropy of the system

$$H(S) = -\left(\frac{1+S}{2}\right) \ln\left(\frac{1+S}{2}\right) - \left(\frac{1-S}{2}\right) \ln\left(\frac{1-S}{2}\right), \tag{11}$$

exhibit the same qualitative behavior of the concurrence. If $S = 1$ the qubit is in a pure state, the total state is separable, and $H = C = 0$. If $S = 0$ the qubit is in the maximally mixed state, where its entropy is maximal ($\ln 2$) and it is also maximally entangled (maximal concurrence). The mean entropy of the system (the qubit),

$$\overline{H} = \int_0^1 H(S)\, \mathrm{p}(S)\, dS = \psi(2\, d) - \psi(d) - \frac{1}{d} \xrightarrow[d \gg 1]{} \ln 2 - \frac{1}{d}, \tag{12}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function, tends to $\ln 2$ (entropy of the maximally mixed state) when $N \to \infty$.

Conclusions

In this manuscript the distribution function for a qubit out of a N+1 qubit system assumed to be in a pure state was calculated. The complete statistics of entanglement can be given if concurrence is used as a measure. It was shown that for $N \to \infty$ the only posible state of every qubit in the N+1 qubit system is the maximally mixed and maximally entangled state.

The authors acknowledge Juan Diego Urbina (Universidad Nacional de Colombia) for fruitful suggestions given in the accomplishment of this work. J. R. M. thanks to Alonso Botero (Universidad de los Andes) for guiding him into the Information World, and to Mauricio Mari'no and Inti Sodemann for helpful discussions.

Referencias

[1] S. Popescu, A. J. Short and A. Winter. *The foundations of statistical mechanics from entanglement: Individual states vs. averages*. arXiv:quant-ph/0511225 v3.

[2] F. Mintert, A. Carvalho, M. Kuś and A. Buchleitner. *Measures and dynamics of entangled states*. arXiv:quant-ph/0505162 v1.